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EIGHTH EDITION

HIGHER ENGINEERING MATHEMATICS

JOHN BIRD

Higher Engineering Mathematics

Now in its eighth edition, *Higher Engineering Mathematics* has helped thousands of students succeed in their exams. Theory is kept to a minimum, with the emphasis firmly placed on problem-solving skills, making this a thoroughly practical introduction to the advanced engineering mathematics that students need to master. The extensive and thorough topic coverage makes this an ideal text for upper-level vocational courses and for undergraduate degree courses. It is also supported by a fully updated companion website with resources for both students and lecturers. It has full solutions to all 2,000 further questions contained in the 277 practice exercises.

John Bird, BSc (Hons), CMath, CEng, CSci, FITE, FIMA, FCollT, is the former Head of Applied Electronics in the Faculty of Technology at Highbury College, Portsmouth, UK. More recently he has combined freelance lecturing and examining, and is the author of over 130 textbooks on engineering and mathematical subjects with worldwide sales of over one million copies. He is currently lecturing at the Defence School of Marine Engineering in the Defence College of Technical Training at HMS Sultan, Gosport, Hampshire, UK.

Why is knowledge of mathematics important in engineering?

A career in any engineering or scientific field will require both basic and advanced mathematics. Without mathematics to determine principles, calculate dimensions and limits, explore variations, prove concepts, and so on, there would be no mobile telephones, televisions, stereo systems, video games, microwave ovens, computers, or virtually anything electronic. There would be no bridges, tunnels, roads, skyscrapers, automobiles, ships, planes, rockets or most things mechanical. There would be no metals beyond the common ones, such as iron and copper, no plastics, no synthetics. In fact, society would most certainly be less advanced without the use of mathematics throughout the centuries and into the future.

Electrical engineers require mathematics to design, develop, test, or supervise the manufacturing and installation of electrical equipment, components, or systems for commercial, industrial, military, or scientific use.

Mechanical engineers require mathematics to perform engineering duties in planning and designing tools, engines, machines, and other mechanically functioning equipment; they oversee installation, operation, maintenance, and repair of such equipment as centralised heat, gas, water, and steam systems.

Aerospace engineers require mathematics to perform a variety of engineering work in designing, constructing, and testing aircraft, missiles, and spacecraft; they conduct basic and applied research to evaluate adaptability of materials and equipment to aircraft design and manufacture and recommend improvements in testing equipment and techniques.

Nuclear engineers require mathematics to conduct research on nuclear engineering problems or apply principles and theory of nuclear science to problems concerned with release, control, and utilisation of nuclear energy and nuclear waste disposal.

Petroleum engineers require mathematics to devise methods to improve oil and gas well production and determine the need for new or modified tool designs; they oversee drilling and offer technical advice to achieve economical and satisfactory progress.

Industrial engineers require mathematics to design, develop, test, and evaluate integrated systems for managing industrial production processes, including human work factors, quality control, inventory control, logistics and material flow, cost analysis, and production co-ordination.

Environmental engineers require mathematics to design, plan, or perform engineering duties in the prevention, control, and remediation of environmental health hazards, using various engineering disciplines; their work may include waste treatment, site remediation, or pollution control technology.

Civil engineers require mathematics in all levels in civil engineering – structural engineering, hydraulics and geotechnical engineering are all fields that employ mathematical tools such as differential equations, tensor analysis, field theory, numerical methods and operations research.

Knowledge of mathematics is therefore needed by each of the engineering disciplines listed above.

It is intended that this text – *Higher Engineering Mathematics* – will provide a step-by-step approach to learning fundamental mathematics needed for your engineering studies.

Higher Engineering Mathematics

Eighth Edition

John Bird

Routledge
Taylor & Francis Group
LONDON AND NEW YORK

Eighth edition published 2017
by Routledge
2 Park Square, Milton Park, Abingdon, Oxon OX14 4RN

and by Routledge
711 Third Avenue, New York, NY 10017

Routledge is an imprint of the Taylor & Francis Group, an informa business

First edition published by Elsevier 1993
Seventh edition published by Routledge 2014

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication Data
Names: Bird, J. O.Title: Higher engineering mathematics / John Bird.
Description: 8th ed. | Boca Raton : CRC Press, 2017. | Includes index.
Identifiers: LCCN 2016038671 | ISBN 9781138673571
Subjects: LCSH: Engineering mathematics.
Classification: LCC TA330 .B52 2017 | DDC 620.001/51–dc23LC
record available at https://lccn.loc.gov/2016038671

ISBN: 978-1-138-67357-1 (pbk)
ISBN: 978-1-315-26502-5 (ebk)

Typeset in Times by
Servis Filmsetting Ltd, Stockport, Cheshire

Visit the companion website: www.routledge.com/cw/bird

To Sue

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Preface

This **eighth edition of *Higher Engineering Mathematics*** covers essential mathematical material suitable for students studying **Degrees, Foundation Degrees, and Higher National Certificate and Diploma courses in Engineering disciplines.**

The text has been conveniently divided into the following **fourteen convenient categories**: number and algebra, geometry and trigonometry, graphs, complex numbers, matrices and determinants, vector geometry, introduction to calculus, further differential calculus, further integral calculus, further differential equations, statistics and probability, Laplace transforms, Fourier series and z-transforms.

Increasingly, **difficulty in understanding algebra** is proving a problem for many students as they commence studying engineering courses. Inevitably there are a lot of formulae and calculations involved with engineering studies that require a sound grasp of algebra. On the website www.routledge.com/cw/bird/ is a document which offers **a quick revision of the main areas of algebra** essential for further study, i.e. basic algebra, simple equations, transposition of formulae, simultaneous equations and quadratic equations.

In this new edition, all of the chapters of the previous edition are included, plus one extra, but the order of presenting some of the calculus chapters has been changed. **New material** has been added on the introduction to numbering systems, Bayes' theorem in probability, the comparison of numerical methods and z-transforms.

The **primary aim of the material in this text** is to provide the fundamental analytical and underpinning knowledge and techniques needed to successfully complete scientific and engineering principles modules of Degree, Foundation Degree and Higher National Engineering programmes. The material has been designed to enable students to use techniques learned for the analysis, modelling and solution of realistic engineering problems at Degree and Higher National level. It also aims to provide some of the more advanced knowledge required for those wishing to pursue careers in mechanical engineering, aeronautical engineering, electrical and electronic engineering, communications engineering, systems engineering and all variants of control engineering.

In ***Higher Engineering Mathematics 8th Edition***, theory is introduced in each chapter by a full outline of essential definitions, formulae, laws, procedures, etc; **problem solving** is extensively used to establish and exemplify the theory. It is intended that readers will gain real understanding through seeing problems solved and then through solving similar problems themselves.

Access to **software packages** such as Maple, Mathematica and Derive, or a graphics calculator, will enhance understanding of some of the topics in this text.

Each topic considered in the text is presented in a way that assumes in the reader only knowledge attained in BTEC National Certificate/Diploma, or similar, in an Engineering discipline.

Higher Engineering Mathematics 8th Edition* provides a follow-up to *Engineering Mathematics 8th Edition.

This textbook contains over **1050 worked problems**, followed by nearly **2000 further problems (with answers),** arranged within **277 Practice Exercises**. Some **552 line diagrams** further enhance understanding.

W**orked solutions** to all 2000 of the further problems have been prepared and can be **accessed free by students and staff via the website** www.routledge.com/cw/bird/

At the end of the text, a list of **Essential Formulae** is included for convenience of reference.

At intervals throughout the text are some **21 Revision Tests** to check understanding. For example, Revision Test 1 covers the material in chapters 1 to 5, Revision Test 2 covers the material in chapters 6 to 8, Revision Test 3 covers the material in chapters 9 to 11, and so on. An **Instructor's Manual**, containing full solutions to the Revision Tests, is available free to lecturers/instructors via the website (see below).

'Learning by example' is at the heart of *Higher Engineering Mathematics 8th Edition*.

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Free Web downloads

The following support material is available from www.routledge.com/cw/bird/

For Students:

1. Full solutions to all 2000 further questions contained in the 277 Practice Exercises
2. Revision of some important algebra topics
3. A list of Essential Formulae
4. Information on 32 Mathematicians/Engineers mentioned in the text

For Lecturers/Instructors:

1. Full solutions to all 2000 further questions contained in the 277 Practice Exercises
2. Revision of some important algebra topics
3. Full solutions and marking scheme for each of the 21 Revision Tests; also, each test may be downloaded for distribution to students. In addition, solutions to the Revision Test given in the 'Revision of Algebra Topics' is also included.
4. A list of Essential Formulae
5. Information on 32 Mathematicians/Engineers mentioned in the text
6. All 552 illustrations used in the text may be downloaded for use in PowerPoint presentations

Syllabus guidance

This textbook is written for **undergraduate engineering degree and foundation degree courses**; however, it is also most appropriate for **BTEC levels 4 and 5 HNC/D studies in engineering** and three syllabuses are covered. The appropriate chapters for these three syllabuses are shown in the table below.

Chapter		**Analytical Methods for Engineers**	**Further Analytical Methods for Engineers**	**Advanced Mathematics for Engineering**
1.	Algebra	×		
2.	Partial fractions	×		
3.	Logarithms	×		
4.	Exponential functions	×		
5.	Inequalities			
6.	Arithmetic and geometric progressions	×		
7.	The binomial series	×		
8.	Maclaurin's series	×		
9.	Solving equations by iterative methods		×	
10.	Binary, octal and hexadecimal		×	
11.	Boolean algebra and logic circuits		×	
12.	Introduction to trigonometry	×		
13.	Cartesian and polar co-ordinates	×		
14.	The circle and its properties	×		
15.	Trigonometric waveforms	×		
16.	Hyperbolic functions	×		
17.	Trigonometric identities and equations	×		
18.	The relationship between trigonometric and hyperbolic functions	×		
19.	Compound angles	×		
20.	Functions and their curves		×	
21.	Irregular areas, volumes and mean value of waveforms		×	
22.	Complex numbers		×	
23.	De Moivre's theorem		×	
24.	The theory of matrices and determinants		×	
25.	Applications of matrices and determinants		×	

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Advanced Mathematics for Engineering
26.	Vectors		×	
27.	Methods of adding alternating waveforms		×	
28.	Scalar and vector products		×	
29.	Methods of differentiation	×		
30.	Some applications of differentiation	×		
31.	Standard integration	×		
32.	Some applications of integration	×		
33.	Introduction to differential equations		×	
34.	Differentiation of parametric equations			
35.	Differentiation of implicit functions	×		
36.	Logarithmic differentiation	×		
37.	Differentiation of hyperbolic functions	×		
38.	Differentiation of inverse trigonometric and hyperbolic functions	×		
39.	Partial differentiation			×
40.	Total differential, rates of change and small changes			×
41.	Maxima, minima and saddle points for functions of two variables			×
42.	Integration using algebraic substitutions	×		
43.	Integration using trigonometric and hyperbolic substitutions	×		
44.	Integration using partial fractions	×		
45.	The t = tan $\theta/2$ substitution			
46.	Integration by parts	×		
47.	Reduction formulae	×		
48.	Double and triple integrals			
49.	Numerical integration		×	
50.	Homogeneous first-order differential equations			
51.	Linear first-order differential equations		×	
52.	Numerical methods for first-order differential equations		×	×
53.	Second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$		×	

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Advanced Mathematics for Engineering
54.	Second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$		×	
55.	Power series methods of solving ordinary differential equations			×
56.	An introduction to partial differential equations			×
57.	Presentation of statistical data	×		
58.	Measures of central tendency and dispersion	×		
59.	Probability	×		
60.	The binomial and Poisson distributions	×		
61.	The normal distribution	×		
62.	Linear correlation	×		
63.	Linear regression	×		
64.	Sampling and estimation theories	×		
65.	Significance testing	×		
66.	Chi-square and distribution-free tests	×		
67.	Introduction to Laplace transforms			×
68.	Properties of Laplace transforms			×
69.	Inverse Laplace transforms			×
70.	The Laplace transform of the Heaviside function			
71.	Solution of differential equations using Laplace transforms			×
72.	The solution of simultaneous differential equations using Laplace transforms			×
73.	Fourier series for periodic functions of period 2π			×
74.	Fourier series for non-periodic functions over range 2π			×
75.	Even and odd functions and half-range Fourier series			×
76.	Fourier series over any range			×
77.	A numerical method of harmonic analysis			×
78.	The complex or exponential form of a Fourier series			×
79.	An introduction to z-transforms			

Section A

Number and algebra

Chapter 1

Algebra

***Why it is important to understand:* Algebra, polynomial division and the factor and remainder theorems**

It is probably true to say that there is no branch of engineering, physics, economics, chemistry or computer science which does not require the understanding of the basic laws of algebra, the laws of indices, the manipulation of brackets, the ability to factorise and the laws of precedence. This then leads to the ability to solve simple, simultaneous and quadratic equations which occur so often. The study of algebra also revolves around using and manipulating polynomials. Polynomials are used in engineering, computer programming, software engineering, in management, and in business. Mathematicians, statisticians and engineers of all sciences employ the use of polynomials to solve problems; among them are aerospace engineers, chemical engineers, civil engineers, electrical engineers, environmental engineers, industrial engineers, materials engineers, mechanical engineers and nuclear engineers. The factor and remainder theorems are also employed in engineering software and electronic mathematical applications, through which polynomials of higher degrees and longer arithmetic structures are divided without any complexity. The study of algebra, equations, polynomial division and the factor and remainder theorems is therefore of some considerable importance in engineering.

At the end of this chapter, you should be able to:

- understand and apply the laws of indices
- understand brackets, factorisation and precedence
- transpose formulae and solve simple, simultaneous and quadratic equations
- divide algebraic expressions using polynomial division
- factorise expressions using the factor theorem
- use the remainder theorem to factorise algebraic expressions

1.1 Introduction

In this chapter, polynomial division and the factor and remainder theorems are explained (in Sections 1.4 to 1.6). However, before this, some essential algebra revision on basic laws and equations is included.
For further algebra revision, go to the website: www.routledge.com/cw/bird

1.2 Revision of basic laws

(a) Basic operations and laws of indices

The **laws of indices** are:

(i) $a^m \times a^n = a^{m+n}$ (ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{m\times n}$ (iv) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$ (vi) $a^0 = 1$

Problem 1. Evaluate $4a^2bc^3 - 2ac$ when $a = 2$, $b = \frac{1}{2}$ and $c = 1\frac{1}{2}$

$$4a^2bc^3 - 2ac = 4(2)^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3 - 2(2)\left(\frac{3}{2}\right)$$

$$= \frac{4 \times 2 \times 2 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2} - \frac{12}{2}$$

$$= 27 - 6 = \mathbf{21}$$

Problem 2. Multiply $3x + 2y$ by $x - y$

$$\begin{array}{lr} & 3x + 2y \\ & x - y \\ \hline \text{Multiply by } x \rightarrow & 3x^2 + 2xy \\ \text{Multiply by } -y \rightarrow & -3xy - 2y^2 \\ \hline \text{Adding gives:} & \mathbf{3x^2 - xy - 2y^2} \\ \hline \end{array}$$

Alternatively,

$$(3x + 2y)(x - y) = 3x^2 - 3xy + 2xy - 2y^2$$

$$= \mathbf{3x^2 - xy - 2y^2}$$

Problem 3. Simplify $\frac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a = 3$, $b = \frac{1}{8}$ and $c = 2$

$$\frac{a^3b^2c^4}{abc^{-2}} = a^{3-1}b^{2-1}c^{4-(-2)} = \boldsymbol{a^2bc^6}$$

When $a = 3$, $b = \frac{1}{8}$ and $c = 2$,

$$a^2bc^6 = (3)^2\left(\frac{1}{8}\right)(2)^6 = (9)\left(\frac{1}{8}\right)(64) = \mathbf{72}$$

Problem 4. Simplify $\frac{x^2y^3 + xy^2}{xy}$

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy}$$

$$= x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$

$$= \boldsymbol{xy^2 + y} \quad \text{or} \quad \boldsymbol{y(xy + 1)}$$

Problem 5. Simplify $\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}}$

$$\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}} = \frac{x^2y^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{5}{2}}y^{\frac{3}{2}}}$$

$$= x^{2+\frac{1}{2}-\frac{5}{2}}y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}$$

$$= x^0y^{-\frac{1}{3}}$$

$$= \boldsymbol{y^{-\frac{1}{3}}} \quad \text{or} \quad \boldsymbol{\frac{1}{y^{\frac{1}{3}}}} \quad \text{or} \quad \boldsymbol{\frac{1}{\sqrt[3]{y}}}$$

Now try the following Practice Exercise

Practice Exercise 1 Basic algebraic operations and laws of indices (Answers on page 856)

1. Evaluate $2ab + 3bc - abc$ when $a = 2$, $b = -2$ and $c = 4$
2. Find the value of $5pq^2r^3$ when $p = \frac{2}{5}$, $q = -2$ and $r = -1$
3. From $4x - 3y + 2z$ subtract $x + 2y - 3z$.
4. Multiply $2a - 5b + c$ by $3a + b$
5. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x = \frac{1}{2}$, $y = 2$ and $z = 3$
6. Evaluate $(a^{\frac{3}{2}}bc^{-3})(a^{\frac{1}{2}}b^{-\frac{1}{2}}c)$ when $a = 3$, $b = 4$ and $c = 2$
7. Simplify $\frac{a^2b + a^3b}{a^2b^2}$
8. Simplify $\frac{(a^3b^{\frac{1}{2}}c^{-\frac{1}{2}})(ab)^{\frac{1}{3}}}{(\sqrt{a^3}\sqrt{b}\,c)}$

(b) Brackets, factorisation and precedence

Problem 6. Simplify $a^2 - (2a - ab) - a(3b + a)$

$$a^2 - (2a - ab) - a(3b + a)$$

$$= a^2 - 2a + ab - 3ab - a^2$$

$$= \boldsymbol{-2a - 2ab} \quad \text{or} \quad \boldsymbol{-2a(1 + b)}$$

Problem 7. Remove the brackets and simplify the expression:

$$2a-[3\{2(4a-b)-5(a+2b)\}+4a]$$

Removing the innermost brackets gives:

$$2a-[3\{8a-2b-5a-10b\}+4a]$$

Collecting together similar terms gives:

$$2a-[3\{3a-12b\}+4a]$$

Removing the 'curly' brackets gives:

$$2a-[9a-36b+4a]$$

Collecting together similar terms gives:

$$2a-[13a-36b]$$

Removing the square brackets gives:

$$2a-13a+36b=\mathbf{-11a+36b} \quad \text{or} \quad \mathbf{36b-11a}$$

Problem 8. Factorise (a) $xy-3xz$ (b) $4a^2+16ab^3$ (c) $3a^2b-6ab^2+15ab$

(a) $xy-3xz=\mathbf{x(y-3z)}$

(b) $4a^2+16ab^3=\mathbf{4a(a+4b^3)}$

(c) $3a^2b-6ab^2+15ab=\mathbf{3ab(a-2b+5)}$

Problem 9. Simplify $3c+2c\times 4c+c\div 5c-8c$

The order of precedence is division, multiplication, addition, and subtraction (sometimes remembered by BODMAS). Hence

$$3c+2c\times 4c+c\div 5c-8c$$
$$=3c+2c\times 4c+\left(\frac{c}{5c}\right)-8c$$
$$=3c+8c^2+\frac{1}{5}-8c$$
$$=\mathbf{8c^2-5c+\frac{1}{5}} \quad \text{or} \quad \mathbf{c(8c-5)+\frac{1}{5}}$$

Problem 10. Simplify $(2a-3)\div 4a+5\times 6-3a$

$$(2a-3)\div 4a+5\times 6-3a$$
$$=\frac{2a-3}{4a}+5\times 6-3a$$
$$=\frac{2a-3}{4a}+30-3a$$
$$=\frac{2a}{4a}-\frac{3}{4a}+30-3a$$
$$=\frac{1}{2}-\frac{3}{4a}+30-3a=\mathbf{30\frac{1}{2}-\frac{3}{4a}-3a}$$

Now try the following Practice Exercise

Practice Exercise 2 Brackets, factorisation and precedence (Answers on page 856)

1. Simplify $2(p+3q-r)-4(r-q+2p)+p$
2. Expand and simplify $(x+y)(x-2y)$
3. Remove the brackets and simplify: $24p-[2\{3(5p-q)-2(p+2q)\}+3q]$
4. Factorise $21a^2b^2-28ab$
5. Factorise $2xy^2+6x^2y+8x^3y$
6. Simplify $2y+4\div 6y+3\times 4-5y$
7. Simplify $3\div y+2\div y-1$
8. Simplify $a^2-3ab\times 2a\div 6b+ab$

1.3 Revision of equations

(a) Simple equations

Problem 11. Solve $4-3x=2x-11$

Since $4-3x=2x-11$ then $4+11=2x+3x$

i.e. $15=5x$ from which, $\boldsymbol{x}=\frac{15}{5}=\mathbf{3}$

Problem 12. Solve

$$4(2a-3)-2(a-4)=3(a-3)-1$$

Removing the brackets gives:

$$8a - 12 - 2a + 8 = 3a - 9 - 1$$

Rearranging gives:

$$8a - 2a - 3a = -9 - 1 + 12 - 8$$

i.e. $3a = -6$

and $\boldsymbol{a} = \frac{-6}{3} = \mathbf{-2}$

Problem 13. Solve $\frac{3}{x-2} = \frac{4}{3x+4}$

By 'cross-multiplying': $3(3x + 4) = 4(x - 2)$

Removing brackets gives: $9x + 12 = 4x - 8$

Rearranging gives: $9x - 4x = -8 - 12$

i.e. $5x = -20$

and $\boldsymbol{x} = \frac{-20}{5} = \mathbf{-4}$

Problem 14. Solve $\left(\frac{\sqrt{t}+3}{\sqrt{t}}\right) = 2$

$$\sqrt{t}\left(\frac{\sqrt{t}+3}{\sqrt{t}}\right) = 2\sqrt{t}$$

i.e. $\sqrt{t} + 3 = 2\sqrt{t}$

and $3 = 2\sqrt{t} - \sqrt{t}$

i.e. $3 = \sqrt{t}$

and $\mathbf{9 = \boldsymbol{t}}$

(c) Transposition of formulae

Problem 15. Transpose the formula $v = u + \frac{ft}{m}$ to make f the subject.

$u + \frac{ft}{m} = v$ from which, $\frac{ft}{m} = v - u$

and $m\left(\frac{ft}{m}\right) = m(v - u)$

i.e. $ft = m(v - u)$

and $\boldsymbol{f = \frac{m}{t}(v - u)}$

Problem 16. The impedance of an a.c. circuit is given by $Z = \sqrt{R^2 + X^2}$. Make the reactance X the subject.

$\sqrt{R^2 + X^2} = Z$ and squaring both sides gives

$R^2 + X^2 = Z^2$, from which,

$X^2 = Z^2 - R^2$ and **reactance** $\boldsymbol{X = \sqrt{Z^2 - R^2}}$

Problem 17. Given that $\frac{D}{d} = \sqrt{\left(\frac{f+p}{f-p}\right)}$ express p in terms of D, d and f.

Rearranging gives: $\sqrt{\left(\frac{f+p}{f-p}\right)} = \frac{D}{d}$

Squaring both sides gives: $\frac{f+p}{f-p} = \frac{D^2}{d^2}$

'Cross-multiplying' gives:

$$d^2(f + p) = D^2(f - p)$$

Removing brackets gives:

$$d^2 f + d^2 p = D^2 f - D^2 p$$

Rearranging gives: $d^2 p + D^2 p = D^2 f - d^2 f$

Factorising gives: $p(d^2 + D^2) = f(D^2 - d^2)$

and $\boldsymbol{p = \frac{f(D^2 - d^2)}{(d^2 + D^2)}}$

Now try the following Practice Exercise

Practice Exercise 3 Simple equations and transposition of formulae (Answers on page 856)

In problems 1 to 4 solve the equations

1. $3x - 2 - 5x = 2x - 4$
2. $8 + 4(x - 1) - 5(x - 3) = 2(5 - 2x)$
3. $\frac{1}{3a-2} + \frac{1}{5a+3} = 0$
4. $\frac{3\sqrt{t}}{1-\sqrt{t}} = -6$
5. Transpose $y = \frac{3(F - f)}{L}$ for f

6. Make l the subject of $t = 2\pi\sqrt{\frac{l}{g}}$
7. Transpose $m = \frac{\mu L}{L + rCR}$ for L
8. Make r the subject of the formula $$\frac{x}{y} = \frac{1+r^2}{1-r^2}$$

(d) Simultaneous equations

Problem 18. Solve the simultaneous equations:

$$7x - 2y = 26 \quad (1)$$

$$6x + 5y = 29 \quad (2)$$

5 × equation (1) gives:

$$35x - 10y = 130 \quad (3)$$

2 × equation (2) gives:

$$12x + 10y = 58 \quad (4)$$

Equation (3) + equation (4) gives:

$$47x + 0 = 188$$

from which, $\boldsymbol{x} = \frac{188}{47} = \mathbf{4}$

Substituting $x = 4$ in equation (1) gives:

$$28 - 2y = 26$$

from which, $28 - 26 = 2y$ and $\mathbf{y = 1}$

Problem 19. Solve

$$\frac{x}{8} + \frac{5}{2} = y \quad (1)$$

$$11 + \frac{y}{3} = 3x \quad (2)$$

8 × equation (1) gives: $x + 20 = 8y$ (3)

3 × equation (2) gives: $33 + y = 9x$ (4)

i.e. $x - 8y = -20$ (5)

and $9x - y = 33$ (6)

8 × equation (6) gives: $72x - 8y = 264$ (7)

Equation (7) − equation (5) gives:

$$71x = 284$$

from which, $\boldsymbol{x} = \frac{284}{71} = \mathbf{4}$

Substituting $x = 4$ in equation (5) gives:

$$4 - 8y = -20$$

from which, $4 + 20 = 8y$ and $\mathbf{y = 3}$

(e) Quadratic equations

Problem 20. Solve the following equations by factorisation:
(a) $3x^2 - 11x - 4 = 0$
(b) $4x^2 + 8x + 3 = 0$

(a) The factors of $3x^2$ are $3x$ and x and these are placed in brackets thus:

$(3x \quad)(x \quad)$

The factors of −4 are +1 and −4 or −1 and +4, or −2 and +2. Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is +1 and −4, i.e.,

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

Thus $(3x + 1)(x - 4) = 0$ hence

either $(3x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $(x - 4) = 0$ i.e. $\boldsymbol{x = 4}$

(b) $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Thus $(2x + 3)(2x + 1) = 0$ hence

either $(2x + 3) = 0$ i.e. $\boldsymbol{x = -\frac{3}{2}}$

or $(2x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{2}}$

Problem 21. The roots of a quadratic equation are $\frac{1}{3}$ and −2. Determine the equation in x.

If $\frac{1}{3}$ and −2 are the roots of a quadratic equation then,

$$(x - \frac{1}{3})(x + 2) = 0$$

i.e. $x^2 + 2x - \frac{1}{3}x - \frac{2}{3} = 0$

i.e. $x^2 + \frac{5}{3}x - \frac{2}{3} = 0$

or $\mathbf{3x^2 + 5x - 2 = 0}$

Problem 22. Solve $4x^2 + 7x + 2 = 0$ giving the answer correct to 2 decimal places.

From the quadratic formula if $ax^2 + bx + c = 0$ then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence if $4x^2 + 7x + 2 = 0$

then $$x = \frac{-7 \pm \sqrt{7^2 - 4(4)(2)}}{2(4)}$$

$$= \frac{-7 \pm \sqrt{17}}{8}$$

$$= \frac{-7 \pm 4.123}{8}$$

$$= \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

i.e. $\boldsymbol{x = -0.36}$ or $\boldsymbol{-1.39}$

Now try the following Practice Exercise

Practice Exercise 4 Simultaneous and quadratic equations (Answers on page 856)

In problems 1 to 3, solve the simultaneous equations

1. $8x - 3y = 51$

   $3x + 4y = 14$

2. $5a = 1 - 3b$

   $2b + a + 4 = 0$

3. $\frac{x}{5} + \frac{2y}{3} = \frac{49}{15}$

   $\frac{3x}{7} - \frac{y}{2} + \frac{5}{7} = 0$

4. Solve the following quadratic equations by factorisation:

   (a) $x^2 + 4x - 32 = 0$

   (b) $8x^2 + 2x - 15 = 0$

5. Determine the quadratic equation in x whose roots are 2 and -5

6. Solve the following quadratic equations, correct to 3 decimal places:

   (a) $2x^2 + 5x - 4 = 0$

   (b) $4t^2 - 11t + 3 = 0$

1.4 Polynomial division

Before looking at long division in algebra let us revise long division with numbers (we may have forgotten, since calculators do the job for us!).

For example, $\frac{208}{16}$ is achieved as follows:

$$\begin{array}{r|l} & 13 \\ \hline 16 & 208 \\ & 16 \\ \hline & 48 \\ & 48 \\ \hline & \cdot\cdot \end{array}$$

(1) 16 divided into 2 won't go

(2) 16 divided into 20 goes 1

(3) Put 1 above the zero

(4) Multiply 16 by 1 giving 16

(5) Subtract 16 from 20 giving 4

(6) Bring down the 8

(7) 16 divided into 48 goes 3 times

(8) Put the 3 above the 8

(9) $3 \times 16 = 48$

(10) $48 - 48 = 0$

Hence $\frac{208}{16} = \mathbf{13}$ exactly

Similarly, $\frac{172}{15}$ is laid out as follows:

$$\begin{array}{r|l} & 11 \\ \hline 15 & 172 \\ & 15 \\ \hline & 22 \\ & 15 \\ \hline & 7 \end{array}$$

Hence $\frac{172}{15} = 11$ remainder 7 or $11 + \frac{7}{15} = \mathbf{11\frac{7}{15}}$

Below are some examples of division in algebra, which in some respects is similar to long division with numbers.

(Note that a **polynomial** is an expression of the form

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$

and **polynomial division** is sometimes required when resolving into partial fractions – see Chapter 2.)

Problem 23. Divide $2x^2 + x - 3$ by $x - 1$

$2x^2 + x - 3$ is called the **dividend** and $x - 1$ the **divisor**. The usual layout is shown below with the dividend and divisor both arranged in descending powers of the symbols.

$$\begin{array}{r l}
 & \quad 2x + 3 \\
x - 1 & \big)\overline{2x^2 + \;\; x - 3} \\
 & \quad \underline{2x^2 - 2x} \\
 & \qquad\qquad 3x - 3 \\
 & \qquad\qquad \underline{3x - 3} \\
 & \qquad\qquad \underline{\;\cdot \quad \cdot\;}
\end{array}$$

Dividing the first term of the dividend by the first term of the divisor, i.e. $\frac{2x^2}{x}$ gives $2x$, which is put above the first term of the dividend as shown. The divisor is then multiplied by $2x$, i.e. $2x(x - 1) = 2x^2 - 2x$, which is placed under the dividend as shown. Subtracting gives $3x - 3$. The process is then repeated, i.e. the first term of the divisor, x, is divided into $3x$, giving $+3$, which is placed above the dividend as shown. Then $3(x - 1) = 3x - 3$, which is placed under the $3x - 3$. The remainder, on subtraction, is zero, which completes the process.

Thus $\mathbf{(2x^2 + x - 3) \div (x - 1) = (2x + 3)}$

[A check can be made on this answer by multiplying $(2x + 3)$ by $(x - 1)$ which equals $2x^2 + x - 3$.]

Problem 24. Divide $3x^3 + x^2 + 3x + 5$ by $x + 1$

$$\begin{array}{r l}
 & \quad (1) \;\; (4) \;\; (7) \\
 & \quad 3x^2 - 2x \;\; + 5 \\
x + 1 & \big)\overline{3x^3 + \;\; x^2 + 3x + 5} \\
 & \quad \underline{3x^3 + 3x^2 \qquad\qquad} \\
 & \qquad\quad -2x^2 + 3x + 5 \\
 & \qquad\quad \underline{-2x^2 - 2x} \\
 & \qquad\qquad\qquad 5x + 5 \\
 & \qquad\qquad\qquad \underline{5x + 5} \\
 & \qquad\qquad\qquad \underline{\;\cdot \quad \cdot\;}
\end{array}$$

(1) x into $3x^3$ goes $3x^2$. Put $3x^2$ above $3x^3$

(2) $3x^2(x + 1) = 3x^3 + 3x^2$

(3) Subtract

(4) x into $-2x^2$ goes $-2x$. Put $-2x$ above the dividend

(5) $-2x(x + 1) = -2x^2 - 2x$

(6) Subtract

(7) x into $5x$ goes 5. Put 5 above the dividend

(8) $5(x + 1) = 5x + 5$

(9) Subtract

Thus $\quad \frac{3x^3 + x^2 + 3x + 5}{x + 1} = \mathbf{3x^2 - 2x + 5}$

Problem 25. Simplify $\frac{x^3 + y^3}{x + y}$

$$\begin{array}{r l}
 & \quad (1) \;\;\; (4) \;\;\; (7) \\
 & \quad x^2 - \; xy \;\; + y^2 \\
x + y & \big)\overline{x^3 + \;\; 0 \;\; + \;\; 0 \;\; + y^3} \\
 & \quad \underline{x^3 + x^2y \qquad\qquad\quad} \\
 & \qquad -x^2y \qquad\quad + y^3 \\
 & \qquad \underline{-x^2y - xy^2 \qquad} \\
 & \qquad\qquad\qquad xy^2 + y^3 \\
 & \qquad\qquad\qquad \underline{xy^2 + y^3} \\
 & \qquad\qquad\qquad \underline{\;\;\cdot \quad \cdot\;\;}
\end{array}$$

(1) x into x^3 goes x^2. Put x^2 above x^3 of dividend

(2) $x^2(x + y) = x^3 + x^2y$

(3) Subtract

(4) x into $-x^2y$ goes $-xy$. Put $-xy$ above dividend

(5) $-xy(x + y) = -x^2y - xy^2$

(6) Subtract

(7) x into xy^2 goes y^2. Put y^2 above dividend

(8) $y^2(x + y) = xy^2 + y^3$

(9) Subtract

Thus

$$\frac{x^3 + y^3}{x + y} = \mathbf{x^2 - xy + y^2}$$

The zeros shown in the dividend are not normally shown, but are included to clarify the subtraction process and to keep similar terms in their respective columns.

Problem 26. Divide $(x^2 + 3x - 2)$ by $(x - 2)$

$$
\begin{array}{r|l}
 & x \;\; + 5 \\
\hline
x - 2 & x^2 + 3x - \;\; 2 \\
 & x^2 - 2x \\
\hline
 & \quad\quad 5x - \;\; 2 \\
 & \quad\quad 5x - 10 \\
\hline
 & \quad\quad\quad\quad\;\; 8 \\
\hline
\end{array}
$$

Hence

$$\frac{x^2 + 3x - 2}{x - 2} = \mathbf{x + 5 + \frac{8}{x - 2}}$$

Problem 27. Divide $4a^3 - 6a^2b + 5b^3$ by $2a - b$

$$
\begin{array}{r|l}
 & 2a^2 - 2ab - \quad b^2 \\
\hline
2a - b & 4a^3 - 6a^2b \quad\quad\quad + 5b^3 \\
 & 4a^3 - 2a^2b \\
\hline
 & \quad\quad -4a^2b \quad\quad\quad + 5b^3 \\
 & \quad\quad -4a^2b + 2ab^2 \\
\hline
 & \quad\quad\quad\quad\quad -2ab^2 + 5b^3 \\
 & \quad\quad\quad\quad\quad -2ab^2 + \;\; b^3 \\
\hline
 & \quad\quad\quad\quad\quad\quad\quad\quad 4b^3 \\
\hline
\end{array}
$$

Thus

$$\frac{4a^3 - 6a^2b + 5b^3}{2a - b}$$

$$= \mathbf{2a^2 - 2ab - b^2 + \frac{4b^3}{2a - b}}$$

Now try the following Practice Exercise

Practice Exercise 5 Polynomial division (Answers on page 856)

1. Divide $(2x^2 + xy - y^2)$ by $(x + y)$
2. Divide $(3x^2 + 5x - 2)$ by $(x + 2)$
3. Determine $(10x^2 + 11x - 6) \div (2x + 3)$
4. Find $\dfrac{14x^2 - 19x - 3}{2x - 3}$
5. Divide $(x^3 + 3x^2y + 3xy^2 + y^3)$ by $(x + y)$
6. Find $(5x^2 - x + 4) \div (x - 1)$
7. Divide $(3x^3 + 2x^2 - 5x + 4)$ by $(x + 2)$
8. Determine $(5x^4 + 3x^3 - 2x + 1)/(x - 3)$

1.5 The factor theorem

There is a simple relationship between the factors of a quadratic expression and the roots of the equation obtained by equating the expression to zero.
For example, consider the quadratic equation
$x^2 + 2x - 8 = 0$
To solve this we may factorise the quadratic expression
$x^2 + 2x - 8$ giving $(x - 2)(x + 4)$
Hence $(x - 2)(x + 4) = 0$
Then, if the product of two numbers is zero, one or both of those numbers must equal zero. Therefore,

either $(x - 2) = 0$, from which, $x = 2$
or $\quad\;\; (x + 4) = 0$, from which, $x = -4$

It is clear, then, that a factor of $(x - 2)$ indicates a root of $+2$, while a factor of $(x + 4)$ indicates a root of -4
In general, we can therefore say that:

a factor of $(x - a)$ corresponds to a root of $x = a$

In practice, we always deduce the roots of a simple quadratic equation from the factors of the quadratic expression, as in the above example. However, we could reverse this process. If, by trial and error, we could determine that $x = 2$ is a root of the equation $x^2 + 2x - 8 = 0$ we could deduce at once that $(x - 2)$ is a factor of the expression $x^2 + 2x - 8$. We wouldn't normally solve quadratic equations this way – but suppose we have to factorise a cubic expression (i.e. one in which the highest power of the variable is 3). A cubic equation might have three simple linear factors and the difficulty of discovering all these factors by trial and error would be considerable. It is to deal with this kind of case that we use the **factor theorem**. This is just a generalised version of what we established above for the quadratic expression. The factor theorem provides a method of factorising any polynomial, $f(x)$, which has simple factors.
A statement of the **factor theorem** says:

'if $x = a$ is a root of the equation
$f(x) = 0$, then $(x - a)$ is a factor of $f(x)$'

The following worked problems show the use of the factor theorem.

Problem 28. Factorise $x^3 - 7x - 6$ and use it to solve the cubic equation $x^3 - 7x - 6 = 0$.

Let $f(x) = x^3 - 7x - 6$

If $x = 1$, then $f(1) = 1^3 - 7(1) - 6 = -12$

If $x = 2$, then $f(2) = 2^3 - 7(2) - 6 = -12$

If $x = 3$, then $f(3) = 3^3 - 7(3) - 6 = 0$

If $f(3) = 0$, then $(x - 3)$ is a factor – from the factor theorem.
We have a choice now. We can divide $x^3 - 7x - 6$ by $(x - 3)$ or we could continue our 'trial and error' by substituting further values for x in the given expression – and hope to arrive at $f(x) = 0$
Let us do both ways. Firstly, dividing out gives:

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ \hline x - 3 & x^3 - 0 \quad - 7x - 6 \\ & x^3 - 3x^2 \\ \hline & \quad 3x^2 - 7x - 6 \\ & \quad 3x^2 - 9x \\ \hline & \qquad 2x - 6 \\ & \qquad 2x - 6 \\ \hline & \qquad \cdot \quad \cdot \end{array}$$

Hence $\dfrac{x^3 - 7x - 6}{x - 3} = x^2 + 3x + 2$

i.e. $x^3 - 7x - 6 = (x - 3)(x^2 + 3x + 2)$
$x^2 + 3x + 2$ factorises 'on sight' as $(x + 1)(x + 2)$.
Therefore

$$\mathbf{x^3 - 7x - 6 = (x - 3)(x + 1)(x + 2)}$$

A second method is to continue to substitute values of x into $f(x)$.
Our expression for $f(3)$ was $3^3 - 7(3) - 6$. We can see that if we continue with positive values of x the first term will predominate such that $f(x)$ will not be zero.
Therefore let us try some negative values for x. Therefore $f(-1) = (-1)^3 - 7(-1) - 6 = 0$; hence $(x + 1)$ is a factor (as shown above). Also $f(-2) = (-2)^3 - 7(-2) - 6 = 0$; hence $(x + 2)$ is a factor (also as shown above).

To solve $x^3 - 7x - 6 = 0$, we substitute the factors, i.e.

$$(x - 3)(x + 1)(x + 2) = 0$$

from which, $\mathbf{x = 3, x = -1}$ **and** $\mathbf{x = -2}$
Note that the values of x, i.e. 3, -1 and -2, are all factors of the constant term, i.e. 6. This can give us a clue as to what values of x we should consider.

Problem 29. Solve the cubic equation $x^3 - 2x^2 - 5x + 6 = 0$ by using the factor theorem.

Let $f(x) = x^3 - 2x^2 - 5x + 6$ and let us substitute simple values of x like 1, 2, 3, -1, -2, and so on.

$$f(1) = 1^3 - 2(1)^2 - 5(1) + 6 = 0,$$
hence $(x - 1)$ is a factor

$$f(2) = 2^3 - 2(2)^2 - 5(2) + 6 \neq 0$$
$$f(3) = 3^3 - 2(3)^2 - 5(3) + 6 = 0,$$
hence $(x - 3)$ is a factor

$$f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 \neq 0$$
$$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0,$$
hence $(x + 2)$ is a factor

Hence $x^3 - 2x^2 - 5x + 6 = (x - 1)(x - 3)(x + 2)$
Therefore if $x^3 - 2x^2 - 5x + 6 = 0$
then $(x - 1)(x - 3)(x + 2) = 0$
from which, $\mathbf{x = 1, x = 3}$ **and** $\mathbf{x = -2}$
Alternatively, having obtained one factor, i.e. $(x - 1)$ we could divide this into $(x^3 - 2x^2 - 5x + 6)$ as follows:

$$\begin{array}{r|l} & x^2 - \ x \ - 6 \\ \hline x - 1 & x^3 - 2x^2 - 5x + 6 \\ & x^3 - \ x^2 \\ \hline & \quad - x^2 - 5x + 6 \\ & \quad - x^2 + \ x \\ \hline & \qquad - 6x + 6 \\ & \qquad - 6x + 6 \\ \hline & \qquad \cdot \quad \cdot \end{array}$$

Hence $x^3 - 2x^2 - 5x + 6$

$$= (x - 1)(x^2 - x - 6)$$

$$\mathbf{= (x - 1)(x - 3)(x + 2)}$$